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A METHOD FOR THE ESTIMATION OF PLASTIC BUCKLING LOADS†

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Abstract—Certain elastic–plastic buckling problems require the solution of an appropriate incremental or "rate" boundary-value problem in order that physically meaningful results may be obtained. In this paper, it is shown that a recent general variational theorem by Neale[8] may be advantageous for the approximate solution of such problems. As an example, the buckling of elastic–plastic cylindrical shells under torsion is analyzed, wherein the material is assumed to obey the incremental theory of plasticity, and the effects of initial imperfections in geometry are taken into account.

INTRODUCTION

In [1, 2] Onat and Drucker have clearly established that observed elastic–plastic buckling loads are generally maximum loads corresponding to certain imperfections in geometry; and that to properly determine the critical load for plastic buckling, a solution of the relevant boundary-value problem for an initially imperfect solid is often required. Furthermore, theoretical considerations as well as experimental evidence indicate that incremental (or rate) stress–strain relations are necessary for a rigourous description of plastic behaviour. Consequently, the "rate problem" becomes appropriate for the investigation of such plastic buckling problems. From the formulation of this rate boundary-value problem (for incremental deformation superimposed on finite deformation) the pertinent load-deformation history of the solid can be obtained in a step-wise manner.

Unfortunately, the solution of typical boundary-value problems encountered is usually a formidable task. As a result, most buckling analyses to date (e.g.[3–5]) have employed the relatively simpler "bifurcation-of-equilibrium" approach; that is, the classical condition of non-uniqueness of solution to the rate boundary-value problem for a geometrically-perfect solid.

Although the bifurcation-of-equilibrium criterion furnishes adequate results for certain buckling problems[3–5], there nevertheless remains a number of cases for which this classical approach must be abandoned and appropriate maximum loads determined. For example, a realistic assessment of the influence of prebuckling deformation, imperfections and boundary conditions on plastic buckling loads[6] generally involves obtaining relevant load-displacement characteristics, and is thus beyond the realm of simple bifurcation theory. Similarly, for the problem of the elastic–plastic buckling of cylindrical shells under torsion[7]

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it can be shown that, if the incremental theory of plasticity is employed, one must consider small initial imperfections in geometry and determine corresponding maximum loads in order to obtain results which correlate well with experimental data.

The purpose of this paper is to demonstrate that a recent general variational principle by Neale for the rate boundary-value problem[8], may be advantageous for the approximate analysis of such elastic–plastic buckling problems. In particular, the example of the torsional buckling of a cylindrical shell discussed above is investigated using the incremental theory of plasticity. Although Gerard[9] and Lee and Ades[10] have previously examined this problem, the physically untenable deformation theory of plasticity is incorporated in their analyses. An analysis based on the more rigourous incremental theory, and accounting for the effects of imperfections, is apparently still unavailable in the literature and is thus presented herein.

LAGRANGIAN FORMULATION OF THE RATE PROBLEM

For the present treatment of plastic buckling problems, it is convenient to express the relevant variables and formulate the fundamental field equations with reference to the initial (undeformed) configuration of the body. Consequently, the Lagrangian (material) description is employed throughout this analysis.

In this description the appropriate measure of the state of deformation is Green's strain tensor, given by[11, 12]

$$E_{ij} = \tfrac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{,j}^{\;k}) \tag{1}$$

where u_i are the components of the displacement vector and a comma denotes covariant differentiation with respect to the initial (material) coordinates a^i. Subsequent deformation of the body from its current (deformed) configuration can be described by the increment of (1), that is Green's strain rate:

$$\dot{E}_{ij} = \tfrac{1}{2}(v_{i,j} + v_{j,i} + v_{k,i}u_{,j}^{\;k} + u_{k,i}v_{,j}^{\;k}) \tag{2}$$

in which $v_i = \dot{u}_i$ represent the velocity components and a rate or increment (dotted quantity) in this description is obtained by a simple partial differentiation with respect to a monotonically increasing variable, such as time.

A suitable stress tensor used in this formulation is the symmetric Kirchhoff stress τ^{ij} which is defined such that

$$F^j = n_i^0(\tau^{ij} + \tau^{ik}u_{,k}^{\;j}) \tag{3}$$

represents the "nominal" traction on an element having an area dS^0 and unit normal n_i^0 in the undeformed state. The components of the load vector on this element of area are then given by

$$dP^j = F^j \, dS^0. \tag{4}$$

In the absence of body forces, the equations of equilibrium for the current configuration are[11, 12]

$$(\tau^{ij} + \tau^{ik}u_{,k}^{\;j})_{,i} = 0. \tag{5}$$

In rate form, the equations for continuing (incremental) equilibrium thus become

$$(\dot{\tau}^{ij} + \dot{\tau}^{ik}u_{,k}^{\;j} + \tau^{ik}v_{,k}^{\;j})_{,i} = 0. \tag{6}$$

In the stepwise solution of the relevant rate equations, it is supposed that the stress distribution, state of deformation and material parameters have already been determined for the current configuration of equilibrium. The typical boundary-value problem then consists of determining the increments or rates of change of these quantities when velocity components v_j^* are prescribed on part of the original boundary surface S_v^0 of the body and nominal traction rates $\dot{F}^{j*}$ are specified on the remainder S_F^0. Thus the boundary conditions for the rate problem are

$$v_j = v_j^* \quad \text{on} \quad S_v^0 \tag{7}$$

$$\dot{F}^j = n_i^0(\dot{\tau}^{ij} + \dot{\tau}^{ik}u_{,k}{}^j + \tau^{ik}v_{,k}{}^j) = \dot{F}^{j*} \quad \text{on} \quad S_F^0 \tag{8}$$

where an asterisk denotes a prescribed quantity.

CONSTITUTIVE LAW

Since an equivalent variational formulation[8] of the rate problem is to be employed in the following, we consider only that class of elastic–plastic strain-hardening solids for which Green's strain rate and Kirchhoff's stress rate are derivable from potential functions. Consequently, the constitutive relations are postulated as follows[6, 8]

$$\dot{E}_{ij} = \frac{\partial W(\dot{\tau})}{\partial \dot{\tau}^{ij}} \text{ or } \dot{\tau}^{ij} = \frac{\partial W(v)}{\partial \dot{E}_{ij}} \tag{9}$$

where $W(\dot{\tau})$ and $W(v)$ are homogeneous functions of degree two in $\dot{\tau}^{ij}$ and $v_{i,j}$ respectively. These rate potentials are related through the Legendre transformation[8]

$$W(v) + W(\dot{\tau}) = \dot{\tau}^{ij}\dot{E}_{ij}. \tag{10}$$

For an isotropic solid characterized by linear elastic response combined with isotropic-hardening J_2 (von Mises) incremental theory, the constitutive equations (9) become[6]

$$\dot{E}_{ij} = \frac{1}{E}[(1+\nu)g_{ik}g_{jl} - \nu g_{ij}g_{kl}]\dot{\tau}^{kl} + G\tau_{ij}'\tau_{kl}'\dot{\tau}^{kl} \tag{11}$$

where

$$\tau^{ij'} = \tau^{ij} - \tfrac{1}{3}g^{ij}g_{kl}\tau^{kl}. \tag{12}$$

In these expressions E and ν are, respectively, the modulus of elasticity and Poisson's ratio, g_{ij} denotes the metric tensor, $\tau^{ij'}$ represents the deviator of Kirchhoff's stress tensor and G is a scalar measure of the current rate of hardening. If it is assumed that the relation between G and J_2 obtained from an uniaxial tension test remains valid for multiaxial states of stress, then[1, 6]

$$G = \begin{cases} \dfrac{3}{4J_2}\left[\dfrac{1}{E^T} - \dfrac{1}{E}\right] & \text{for} \quad \dot{J}_2 > 0 \\ 0 & \ldots \quad \dot{J}_2 \le 0 \end{cases} \tag{13}$$

where E^T represents the tangent modulus (expressed as a function of J_2) and J_2 is given by

$$J_2 = \tfrac{1}{2}g_{ik}g_{jl}\tau^{ij'}\tau^{kl'}. \tag{14}$$

In a concise form, the constitutive relations (11) can be expressed as follows

$$\dot{E}_{ij} = c_{ijkl}\dot{\tau}^{kl}. \tag{15}$$

As a result, the corresponding expression for the rate potential becomes

$$W(\dot{\tau}) = \tfrac{1}{2}\dot{\tau}^{ij}\dot{E}_{ij} = \tfrac{1}{2}c_{ijkl}\dot{\tau}^{ij}\dot{\tau}^{kl}. \tag{16}$$

VARIATIONAL METHOD FOR PLASTIC BUCKLING

A general variational theorem[8] has recently been established for the rate boundary-value problem described by equations (6–9). This theorem, which permits independent variations of both the stress rates $\dot{\tau}^{ij}$ and velocities v_j, is such that the rate equations of equilibrium (6) as well as the constitutive relations (9) are Euler (differential) equations of the variational principle while all boundary conditions for traction rates (8) and velocities (7) are the natural boundary conditions. Hence, this general variational principle for finite deformations of elastic–plastic bodies is somewhat analogous to Reissner's principle[13, 14] in elasticity.

In terms of the foregoing formulation of the rate problem, the general variational principle [8] states that the system of differential equations (6–9) is equivalent to the variational problem

$$\delta I^0 = 0 \tag{17}$$

where

$$I^0 = \int_{V_0} [\dot{\tau}^{ij}\dot{E}_{ij} + \tfrac{1}{2}\tau^{ij}v_{,i}{}^{k}v_{k,j} - W(\dot{\tau})]\,\mathrm{d}V_0 - \int_{S_F{}^0} \dot{F}^{j*}v_j\,\mathrm{d}S^0 - \int_{S_v{}^0} \dot{F}^j(v_j - v_j{}^*)\,\mathrm{d}S^0. \tag{18}$$

That is, the solution of the rate problem is such that the functional I^0 is stationary in the class of continuous stress-rate and velocity fields.

Since the variations of both v_j and $\dot{\tau}^{ij}$ in (18) are arbitrary, this general variational principle therefore allows *independent* selection of both the stress-rate distributions and velocity fields in the approximate solution of given boundary-value problems in finite plasticity; and in particular, it provides a general basis for the estimation of elastic–plastic buckling loads.

As shown in [8], the Legendre transformation (10), together with the restriction that the admissible velocity fields satisfy the prescribed boundary conditions (7), gives

$$I^0 = J^0(v) = \int_{V_0} [W(v) + \tfrac{1}{2}\tau^{ij}v_{,i}{}^{k}v_{k,j}]\,\mathrm{d}V_0 - \int_{S_F{}^0} \dot{F}^{j*}v_j\,\mathrm{d}S^0. \tag{19}$$

This specialized form of I^0, where only the velocity components are varied, has previously been given by Murphy and Lee[6] and applied to the problem of axially compressed cylindrical shells subject to edge constraints. However, the following sample problem indicates that certain computational simplifications arise when the general form (18) is employed, in which both stress-rate fields and velocity distributions are varied independently. Moreover, despite the small number of terms contained in the approximate solution which follows, the numerical results obtained are shown to compare reasonably well with those of [9, 10]. It is perhaps interesting to note that Nemat-Nasser[15] has recently reported a similar observation, in that a general (modified-Reissner) variational method furnished "astonishingly accurate" results for the problem of harmonic waves in elastic composites. In contrast an alternative method, in which only the displacement field was given independent variation, produced extremely poor results[15] unless a much larger number of terms was included in the approximate (Ritz) solutions.

TORSIONAL BUCKLING OF CYLINDRICAL SHELLS

As an example of application of the foregoing variational method for elastic–plastic buckling, we consider the problem of the buckling of a long cylindrical shell twisted axially by a couple M applied at both ends (Fig. 1). The material is assumed to obey the J_2 incremental theory of plasticity.

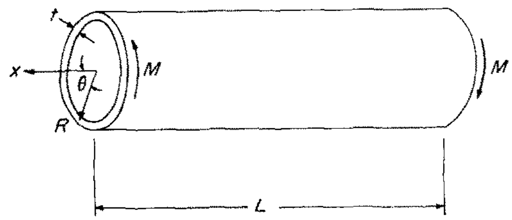

Fig. 1. Cylindrical shell under torsion.

This problem was selected since it presents an interesting paradox, similar to that resolved by Onat and Drucker[1] for the case of the axially compressed cruciform column: If one considers a *geometrically-perfect* cylindrical shell and then determines the critical state for bifurcation of equilibrium according to the von Mises (J_2) incremental theory[7], the critical torque so-obtained is considerably higher than experimental buckling loads. This can be explained by the fact that a primarily elastic response is activated by the ensuing mode of buckling (Fig. 2). The use of Tresca's yield surface (although its corners may affect the critical stress under axial compression[5]) does not provide a better correlation since, for a torsional stress $\tau_{x\theta}$ alone, there is again a primarily elastic response accompanying the superimposed deformation due to buckling (Fig. 2). Yet, the same bifurcation-of-equilibrium approach applied in conjunction with the less rigourous deformation theory[9, 10] furnishes results which adequately reflect plastic behaviour and agree quite well with available experimental data. As for the problem of the cruciform column[1], the solution to this paradox is to take into account the extremely small (and therefore unavoidable) initial imperfections of geometry and to determine the corresponding torque–twist curves, from which more meaningful critical (maximum) couples can be obtained. This approach requires the formulation of the appropriate rate problem, which can subsequently be solved in an approximate manner by employing the variational principle (17) in conjunction with the method of Ritz.

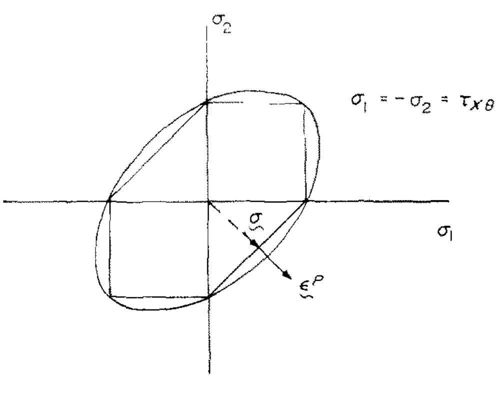

Fig. 2. Stress vector and plastic strain-rate vector coincide for shear $\tau_{x\theta}$ alone.

(a) *Deformation of shell*

For convenience, a cylindrical polar coordinate system is used as a reference where the x-axis coincides with the axis of the cylinder and r, θ refer to the radial and circumferential directions, respectively. Furthermore, the basic relations used throughout the following analysis are henceforth expressed in terms of the physical components[12] of the relevant tensors.

In order to simplify the analysis somewhat, the usual assumptions of thin shell theory[17] are adopted. Consequently, by supposing that normals to the mid-surface of the cylinder remain normal during the process of deformation, we obtain the following expressions for the displacement components[17]

$$u_x = u - z\frac{\partial w}{\partial x}$$
$$u_\theta = v + \frac{z}{R}\left(v - \frac{\partial w}{\partial \theta}\right) \tag{20}$$
$$u_r = w$$

where $u = u(x, \theta)$, $v = v(x, \theta)$, $w = w(x, \theta)$ are the displacements of the mid-surface of the shell in the x, θ, r directions respectively; and

$$z = r - R \tag{21}$$

represents the distance from the middle surface. In view of (1) the corresponding strain components thus become

$$E_{xx} = \frac{\partial u}{\partial x} - z\frac{\partial^2 w}{\partial x^2} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2$$
$$E_{\theta\theta} = \frac{1}{R}\left[w + \frac{\partial v}{\partial \theta} + \frac{z}{R}\left(\frac{\partial v}{\partial \theta} - \frac{\partial^2 w}{\partial \theta^2}\right) + \frac{1}{2R}\left(\frac{\partial w}{\partial \theta}\right)^2\right] \tag{22}$$
$$E_{x\theta} = \frac{1}{2}\left[\frac{1}{R}\frac{\partial u}{\partial \theta} + \frac{\partial v}{\partial x} + \frac{z}{R}\left(\frac{\partial v}{\partial x} - 2\frac{\partial^2 w}{\partial x\,\partial \theta}\right) + \frac{1}{R}\frac{\partial w}{\partial x}\frac{\partial w}{\partial \theta}\right]$$

in which only the nonlinear terms involving the products of the derivatives of the radial displacement w have been retained.

From (20), the velocity or incremental–displacement distributions ($v_i = \dot{u}_i$) are readily obtained. Similarly, equations (22) furnish the following expressions for the strain rates:

$$\dot{E}_{xx} = \frac{\partial \dot{u}}{\partial x} - z\frac{\partial^2 \dot{w}}{\partial x^2} + \frac{\partial w}{\partial x}\frac{\partial \dot{w}}{\partial x}$$
$$\dot{E}_{\theta\theta} = \frac{1}{R}\left[\dot{w} + \frac{\partial \dot{v}}{\partial \theta} + \frac{z}{R}\left(\frac{\partial \dot{v}}{\partial \theta} - \frac{\partial^2 \dot{w}}{\partial \theta^2}\right) + \frac{1}{R}\frac{\partial w}{\partial \theta}\frac{\partial \dot{w}}{\partial \theta}\right] \tag{23}$$
$$\dot{E}_{x\theta} = \frac{1}{2}\left[\frac{1}{R}\frac{\partial \dot{u}}{\partial \theta} + \frac{\partial \dot{v}}{\partial x} + \frac{z}{R}\left(\frac{\partial \dot{v}}{\partial x} - 2\frac{\partial^2 \dot{w}}{\partial x\,\partial \theta}\right) + \frac{1}{R}\left(\frac{\partial w}{\partial x}\frac{\partial \dot{w}}{\partial \theta} + \frac{\partial w}{\partial \theta}\frac{\partial \dot{w}}{\partial x}\right)\right].$$

(b) *Constitutive relations for shell*

As a result of the assumption of plane stress, the constitutive relations (11) become (in terms of physical components)

$$\begin{aligned} \dot{E}_{xx} &= c_{11}\dot{\tau}_{xx} + c_{12}\,\dot{\tau}_{\theta\theta} + c_{13}\,\dot{\tau}_{x\theta} \\ \dot{E}_{\theta\theta} &= c_{21}\dot{\tau}_{xx} + c_{22}\,\dot{\tau}_{\theta\theta} + c_{23}\,\dot{\tau}_{x\theta} \\ \dot{E}_{x\theta} &= c_{31}\dot{\tau}_{xx} + c_{32}\,\dot{\tau}_{\theta\theta} + c_{33}\,\dot{\tau}_{x\theta} \end{aligned} \tag{24}$$

where the material coefficients c_{ij} are given by the following expressions

$$\begin{aligned} c_{11} &= \frac{1}{E} + \frac{G}{9}(2\tau_{xx} - \tau_{\theta\theta})^2 \\ c_{12} = c_{21} &= -\frac{\nu}{E} + \frac{G}{9}(2\tau_{xx} - \tau_{\theta\theta})(2\tau_{\theta\theta} - \tau_{xx}) \\ c_{13} = 2c_{31} &= \frac{2G}{3}(2\tau_{xx} - \tau_{\theta\theta})\tau_{x\theta} \\ c_{22} &= \frac{1}{E} + \frac{G}{9}(2\tau_{\theta\theta} - \tau_{xx})^2 \\ c_{23} = 2c_{32} &= \frac{2G}{3}(2\tau_{\theta\theta} - \tau_{xx})\tau_{x\theta} \\ c_{33} &= \frac{1+\nu}{E} + 2G\tau_{x\theta}^{2}; \end{aligned} \tag{25}$$

For plastic loading ($\dot{J}_2 > 0$) the hardening function G is given by (13), i.e.

$$G = \frac{3}{4J_2}\left[\frac{1}{E^T} - \frac{1}{E}\right]$$

in which

$$J_2 = \tfrac{1}{3}(\tau_{xx}^{\ 2} - \tau_{xx}\tau_{\theta\theta} + \tau_{\theta\theta}^{\ 2}) + \tau_{x\theta}^{\ 2}. \tag{26}$$

If it is furthermore assumed that the uniaxial stress–strain curve can be described by a Ramberg–Osgood relation of the form

$$E_{xx} = \frac{\tau_{xx}}{E} + \left(\frac{\tau_{xx}}{E_0}\right)^k \tag{27}$$

where E_0 and k are material constants, then the above expression for G becomes

$$G = \frac{3k[\sqrt{(3J_2)}]^{k-1}}{4J_2 E_0^{\ k}}. \tag{28}$$

APPROXIMATE SOLUTION FOR BUCKLING LOADS

To obtain the appropriate torque–twist curves required for the determination of plastic buckling loads, we formulate the boundary-value problem such that the increment of relative twist between the ends of the shell $\dot{\psi}^*$ is specified and the corresponding incremental couple

$\dot{M}$ is to be calculated. In this case the previous relations for cylindrical shells under torsion furnish the following expression for the functional I^0 (18):

$$I^0 = R\int_0^L \int_{-t/2}^{t/2} \int_0^{2\pi} [\dot{\tau}_{ij}\dot{E}_{ij} + \tfrac{1}{2}\tau_{ij}v_{k,i}v_{k,j} - W(\dot{\tau})]\, d\theta\, dz\, dx - \dot{M}(\dot{\psi} - \dot{\psi}^*) \tag{29}$$

where

$$\dot{\psi}^* = \frac{\dot{\phi}^*}{R} \tag{30}$$

is the rate (increment) of the prescribed rotation at the end $x = L$ of the cylinder relative to the end $x = 0$, and

$$\begin{aligned} \dot{\tau}_{ij}\dot{E}_{ij} &= \dot{\tau}_{xx}\dot{E}_{xx} + 2\dot{\tau}_{x\theta}\dot{E}_{x\theta} + \dot{\tau}_{\theta\theta}\dot{E}_{\theta\theta} \\ \tau_{ij}v_{k,i}v_{k,j} &= \tau_{xx}\left(\frac{\partial\dot{w}}{\partial x}\right)^2 + \frac{2}{R}\tau_{x\theta}\frac{\partial\dot{w}}{\partial x}\frac{\partial\dot{w}}{\partial\theta} + \frac{1}{R^2}\tau_{\theta\theta}\left(\frac{\partial\dot{w}}{\partial\theta}\right)^2 \\ 2W(\dot{\tau}) &= c_{11}\dot{\tau}_{xx}^2 + 2c_{12}\dot{\tau}_{xx}\dot{\tau}_{\theta\theta} + c_{22}\dot{\tau}_{\theta\theta}^2 + 2c_{13}\dot{\tau}_{xx}\dot{\tau}_{x\theta} + 2c_{23}\dot{\tau}_{\theta\theta}\dot{\tau}_{x\theta} + 2c_{33}\dot{\tau}_{x\theta}^2. \end{aligned} \tag{31}$$

The general variational principle (17) states that the solution to the boundary-value problem is such that I^0 is stationary for arbitrary variations of the stress-rate field and velocity components. Hence, by independently choosing suitable expressions for $\dot{\tau}_{xx}$, $\dot{\tau}_{x\theta}$, $\dot{\tau}_{\theta\theta}$, $\dot{u}(x, \theta)$, $\dot{v}(x, \theta)$ and $\dot{w}(x, \theta)$ the method of Ritz can be applied to obtain an approximate solution for these rate or incremental quantities. The required curve of twisting couple M vs relative end rotation ψ can then be determined by solving this rate problem for successive specified increments $\dot{\psi}^*$. Evidently, the maximum value of M (i.e. $\dot{M} = 0$) on such a curve defines the critical couple M_{cr} for which plastic buckling occurs.

In the following an extremely simplified analysis is presented in which very few terms are employed in the incremental Ritz solution. Although more accurate results could be obtained by increasing the number of terms, we intentionally refrain from refining the present analysis in order to illustrate the somewhat surprising manner in which the buckling loads so-obtained approach existing theoretical values[9], which have previously been shown to agree quite well with experimental results.

In this step-wise determination of the M–ψ curve it is supposed that, for the initial (unloaded) state, the existing imperfection in geometry is of the form

$$w^0 = \delta^0 t \sin\left(\frac{m\pi x}{L} - n\theta\right) \tag{32a}$$

and that all other displacements and stresses are zero, i.e.

$$u^0 = v^0 = \tau_{xx}{}^0 = \tau_{x\theta}{}^0 = \tau_{\theta\theta}{}^0 = 0. \tag{32b}$$

We subsequently specify $\dot{\phi} = \dot{\phi}^*$, and an approximate solution for the corresponding increments of the above quantities from the initial state is then chosen as follows (cf.[16])

$$\begin{aligned}
\dot{u} &= 0 \\
\dot{v} &= \frac{\dot{\phi}^* x}{L} \\
\dot{w} &= \dot{\delta} t \sin\left(\frac{m\pi x}{L} - n\theta\right) \\
\dot{\tau}_{xx} &= \dot{\alpha}_1 \frac{z}{t} \sin\left(\frac{m\pi x}{L} - n\theta\right) \\
\dot{\tau}_{\theta\theta} &= \dot{\alpha}_2 \frac{z}{t} \sin\left(\frac{m\pi x}{L} - n\theta\right) \\
\dot{\tau}_{x\theta} &= \dot{\tau} = \frac{\dot{M}}{2\pi R^2 t}
\end{aligned} \tag{33}$$

where, according to the Ritz technique, the unknown coefficients are determined by substituting these expressions into (29) and using the conditions

$$\frac{\partial I^0}{\partial \dot{\delta}} = \frac{\partial I^0}{\partial \dot{\alpha}_1} = \frac{\partial I^0}{\partial \dot{\alpha}_2} = \frac{\partial I^0}{\partial \dot{\tau}} = 0. \tag{34}$$

As a result of successive applications of the above approximate solution, the state of stress and deformation at a given instant of time thus becomes

$$\begin{aligned}
u &= 0 \\
v &= \frac{\phi^* x}{L} \\
w &= \delta t \sin\left(\frac{m\pi x}{L} - n\theta\right) \\
\tau_{xx} &= \alpha_1 \frac{z}{t} \sin\left(\frac{m\pi x}{L} - n\theta\right) \\
\tau_{\theta\theta} &= \alpha_2 \frac{z}{t} \sin\left(\frac{m\pi x}{L} - n\theta\right) \\
\tau_{x\theta} &= \tau = \frac{M}{2\pi R^2 t}.
\end{aligned} \tag{35}$$

Consequently the approximate solution of the boundary-value problem reduces to the stepwise solution of a set of equations which are linear in the incremental coefficients $\dot{\delta} \ldots \dot{\tau}$ and can be written representatively as follows:

$$[a_{ij}(\delta, \alpha_1, \alpha_2, \tau)] \begin{bmatrix} \dot{\delta} \\ \dot{\alpha}_1 \\ \dot{\alpha}_2 \\ \dot{\tau} \end{bmatrix} = [b_i] \tag{36}$$

subject to the initial conditions:

$$\begin{aligned} \delta &= \delta^0 \\ \alpha_1{}^0 &= \alpha_2{}^0 = \tau^0 = 0. \end{aligned} \tag{37}$$

A perhaps noteworthy feature of this method is that explicity expressions for both stresses as well as displacements [e.g. equations (33) and (35)] are provided, in view of the general variational principle (17). In many cases this allows the functional I^0 in (29) to be integrated analytically, thus eliminating the necessity of performing numerical integrations at each step of the incremental solution. For example, if the exponent in the Ramberg–Osgood relation (27) is $k = 3$ then, from (28), the hardening function G is given by

$$G = \frac{27}{4E_0{}^3} = \text{const.} \tag{38}$$

For that part of the load-deformation history for which plastic loading ($\dot{J}_2 > 0$) occurs throughout the entire shell, the relations (34) or (36) thus become

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{bmatrix} \begin{bmatrix} \dot{\alpha}_1 \\ \dot{\alpha}_2 \\ \dot{\tau} \\ \dot{\delta} \end{bmatrix} = \begin{bmatrix} \dot{\phi}^*/L \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{39}$$

where, in view of (29), (31), (23) and (25):

$$\begin{aligned}
a_{11} &= \frac{G}{36}\tau(2\alpha_1 - \alpha_2), & a_{12} &= \frac{G}{36}\tau(2\alpha_2 - \alpha_1) \\
a_{13} &= 2\left(\frac{1+\nu}{E} + 2G\tau^2\right), & a_{14} &= n\beta\left(\frac{t}{R}\right)^2 \delta \\
a_{21} &= \frac{1}{E} + \frac{G}{80}(2\alpha_1 - \alpha_2)^2, & a_{22} &= -\frac{\nu}{E} + \frac{G}{80}(2\alpha_1 - \alpha_2)(2\alpha_2 - \alpha_1) \\
a_{23} &= \frac{2}{3}G\tau(2\alpha_1 - \alpha_2), & a_{24} &= -\left(\beta\frac{t}{R}\right)^2 \\
a_{31} &= a_{22}, & a_{32} &= \frac{1}{E} + \frac{G}{80}(2\alpha_2 - \alpha_1)^2 \\
a_{33} &= \frac{2}{3}G\tau(2\alpha_2 - \alpha_1), & a_{34} &= -\left(n\frac{t}{R}\right)^2 \\
a_{41} &= \frac{\beta^2}{12}, & a_{42} &= \frac{n^2}{12} \\
a_{43} &= -2n\beta\delta, & a_{44} &= -2n\beta\tau
\end{aligned} \tag{40}$$

$$\beta = \frac{m\pi R}{L}.$$

As a result, the numerical step-wise determination of a load-displacement ($\tau - \phi^*$) curve can be performed directly in this case, and the calculations involved require a minimal amount of computer time. On the other hand, when the variational principle (20) which permits only the incremental-displacement components to be varied is employed in an approximate solution, these computational simplifications do not generally arise since explicit expressions for the current stress distributions [e.g. in (29)] are not usually obtainable, and the time-consuming numerical integrations cannot therefore be avoided.

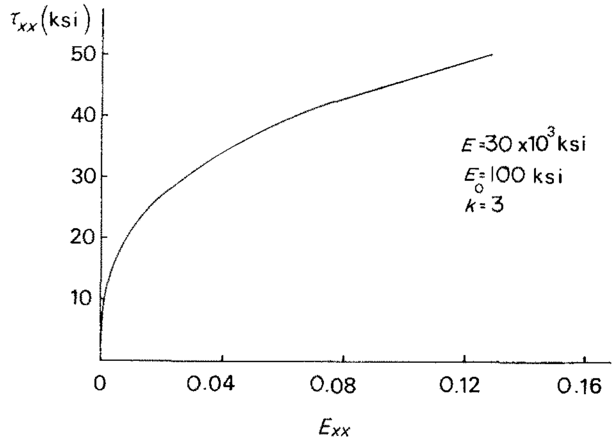

Fig. 3. Kirchhoff stress vs Green's strain curve for uniaxial tension.

RESULTS AND DISCUSSION

Numerical results have been obtained for the case where the material parameters in the Ramberg–Osgood relation (27) are $E = 30 \times 10^3$ ksi, $E_0 = 100$ ksi, $k = 3$ (Fig. 3), and $\nu = 0{\cdot}3$. Typical load–displacement curves (τ vs γ, τ vs δ) corresponding to various initial imperfections δ^0 are shown in Figs. 4 and 5 for geometric parameters $R/t = 20$ and $L/R = 20$.

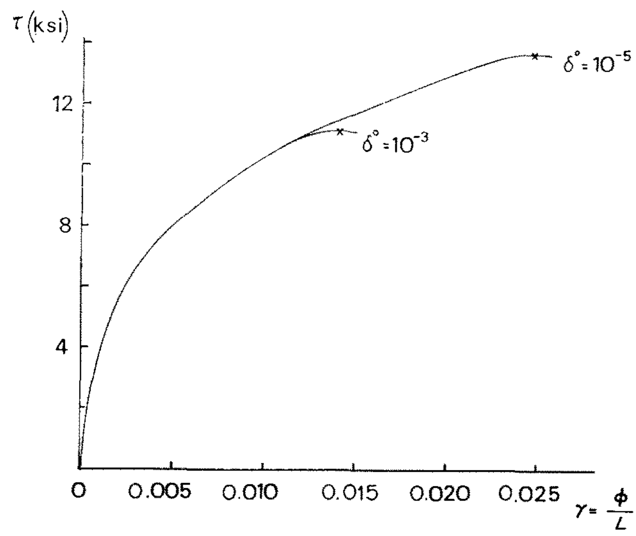

Fig. 4. Torque vs twist curves for $R/t = L/R = 20$.

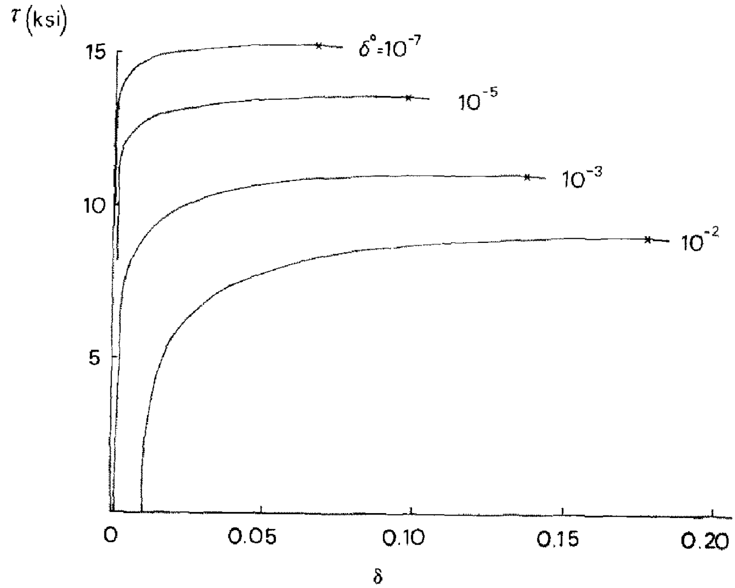

Fig. 5. Torque vs radial displacement curves for $R/t = L/R = 20$.

These curves have been determined by assuming that no unloading occurs in the shell [i.e. equations (40) valid], solving (39) in a step-wise fashion for the incremental quantities $\dot{\alpha}_1 \ldots \dot{\delta}$, and subsequently verifying whether the loading criterion ($\dot{J}_2 > 0$) is satisfied throughout the entire shell. The results indicated that, for each increment up to maximum torsion, plastic loading did in fact occur everywhere. However, unloading took place in some parts of the cylinder beyond the buckling stress τ_{max}. Since the objective of the analysis was to estimate the critical torsional buckling stress ($\tau_{cr} = \tau_{max}$), which an incremental solution of (39) and (40) readily furnished, a "post-buckling" analysis taking unloading into account was not performed for the load-displacement behaviour beyond τ_{cr}. Furthermore, it should also be mentioned that load-displacement curves were obtained for various combinations of the integers m and n in (32), (33) and that the curves represented in Figs. 4 and 5 correspond to that mode of buckling ($n = 2, m = 9$) which produced minimum values of τ_{cr}. As for the axially compressed cylindrical shell[18], the numerical buckling loads obtained were observed to be quite sensitive to the mode of buckling.

The manner in which the critical torsional stress for the above example varies with initial imperfection δ^0 is given in Fig. 6 and Table 1. For the perfect shell the buckling stress, as

Table 1. Effect of initial imperfection

δ^0	τ_{cr} (psi)
0	31,350[7]
10^{-7}	15,280
10^{-5}	13,600
10^{-3}	11,120
10^{-2}	9,030
10^{-1}	6,100

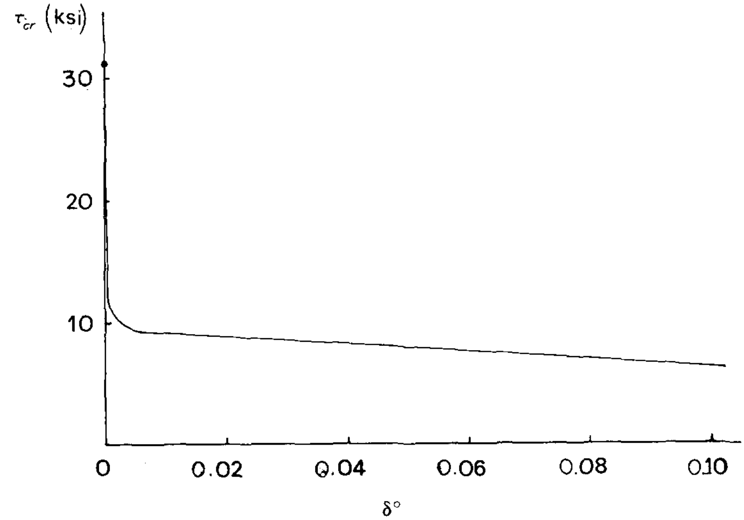

Fig. 6. Variation of buckling stress with initial imperfection.

Table 2. Comparison of results with Gerard's[9]

$\frac{R}{t}$	τ_{cr} (present analysis) $\delta^0 = 0{\cdot}001$	τ_{cr}[9] (deformation theory) $\delta^0 = 0$
20	11,120 psi	10,450 psi
25	8,950	9,270
30	7,490	8,390
35	6,370	7,711
40	5,460	7,150

determined according to the classical bifurcation-of-equilibrium approach[7], is quite high. However, the existence of extremely small, and therefore unavoidable, imperfections (e.g. $\delta^0 = 10^{-7}$) evidently reduce τ_{cr} considerably. Moreover, these results indicate that, although the existence of a small imperfection in geometry is vital for a realistic estimation of the critical couple for plastic buckling, the critical torsion obtained is virtually insensitive to the *amount* of imperfection. Similar observations have been made by Onat and Drucker[1] with regard to the axially compressed cruciform column, which buckles in a twisting mode.

Finally, in Table 2 a comparison is made between the results of the present analysis (with $\delta^0 = 0{\cdot}001$, $L/R = 20$), and those obtained using deformation theory in conjunction with the bifurcation-of-equilibrium criterion[9]. This shows a reasonable agreement for $R/t < 40$, despite the rather simplified approximation (33) used in the present solution.

CONCLUDING REMARKS

The foregoing example suggests that, for certain plastic buckling problems, an application of the general variational principle (17), (18) may result in a fairly simplified analysis. In this case, for example, the problem reduced to the step-wise solution of linear algebraic equations (39), (40) subject to the initial conditions (37). Even though 100–250 increments were employed to determine a particular load–deformation curve, the numerical computations involved required very little computer time.

The results obtained by this method show the critical load for the torsional buckling of cylindrical shells to be very sensitive to small initial imperfections in geometry. This corroborates the conclusions of Onat and Drucker, who clearly showed in [1, 2] that the classical bifurcation-of-equilibrium approach is not always appropriate for plastic buckling problems.

In the above analysis, simplifications arose from the fact that plastic loading existed throughout the entire shell up to the maximum load. Elastic unloading, in fact, first occurred at the maximum load. Furthermore, the numerical results indicated that the functional I^0 (29) first became negative at the maximum load. In view of the relationship between the functionals I^0 (18), J^0 (19) and the uniqueness and stability criteria given in [6], it follows that bifurcation and instability—as well as elastic unloading—occurred simultaneously for this example at τ_{max}. While elastic unloading may precede the maximum load for other problems, simplifications similar to those arising in the present analysis could be obtained by considering the load at which elastic unloading first occurs as an estimate of the critical load [19].

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